

Global Gevrey regularity and analyticity of a two-component shallow water system with Higher-order inertia operators

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Abstract

In this paper, we mainly consider the Gevrey regularity and analyticity of the solution to a generalized two-component shallow water wave system with higher-order inertia operators, namely, $m = (1 - \partial_x^2)^s u$ with $s > 1$. Firstly, we obtain the Gevrey regularity and analyticity for a short time. Secondly, we show the continuity of the data-to-solution map. Finally, we prove the global Gevrey regularity and analyticity in time.

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1 Introduction

In this paper we mainly consider the analyticity and Gevrey regularity of the solution to the following generalized two-component shallow water wave system with higher-order inertia operators[35]:

$$\begin{cases} m_t + um_x + amu_x = \alpha u_x - \kappa \rho \rho_x, & t > 0, x \in \mathbb{R}, \\ \rho_t + u\rho_x + (a-1)u_x\rho = 0, & t > 0, x \in \mathbb{R}, \\ m(t, x) = (1 - \partial_x^2)^s u(t, x), & t \geq 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0, x) = \rho_0(x), & x \in \mathbb{R}, \end{cases} \quad (1.1)$$

where $s > 1$, $a \neq 1$ is a real parameter, α is a constant which represents the vorticity of underlying flow, and $\kappa > 0$ is an arbitrary real parameter. The system (1.1) is the generalization of the two component water wave system with $s = 1$, namely, $m = (1 - \partial_x^2)u = u - u_{xx}$ (see [29],[40] and [52]).

For $\alpha = 0, \rho \equiv 0$, the system (1.1) becomes a family of one-component equations

$$\begin{cases} m_t + um_x + au_xm = 0, & t > 0, x \in \mathbb{R}, \\ m(t, x) = (1 - \partial_x^2)^s u(t, x), & t \geq 0, x \in \mathbb{R}, \\ u(t, x)|_{t=0} = u_0(x). & x \in \mathbb{R}. \end{cases} \quad (1.2)$$

When $s = 1$, the equation (1.2) is called the b-equation. The b-equation possess a number of structural phenomena which are shared by solutions of the family of equations [36, 48, 49]. Recently, some authors were devoted to the study of the Cauchy problem for the b-equation. The local well-posedness of the b-equation was obtained by Escher and Yin in [36] and Gui, Liu and Tian in [47], respectively, on the line and Zhang and Yin in [77] on the circle. It also has global solutions [36, 47, 77] and solutions which blow up in finite time [36, 47, 77]. The uniqueness and existence of global weak solution to the b-equation provided the initial data satisfies certain sign conditions were obtained in [36, 77]. However, there are just two members of this family which are integrable [54]: the Camassa-Holm [6, 5] equation, when $a = 2$, and the Degasperis-Procesi [27] equation, when $a = 3$. The Cauchy problem and initial-boundary value problem for the Camassa-Holm equation have been studied extensively [15, 16, 26, 37, 38, 57, 67, 72]. It has been shown that this equation is locally well-posed [15, 16, 26, 57, 67] for initial data $u_0 \in H^q(\mathbb{R})$, $q > \frac{3}{2}$. More interestingly, it has global strong solutions [11, 15, 16] and also finite time blow-up solutions [11, 14, 15, 16, 18, 26, 57, 67]. On the other hand, it has global weak solutions in $H^1(\mathbb{R})$ [1, 2, 17, 25, 70]. Finite propagation speed and persistence properties for the Camassa-Holm equation have been studied in [13, 51]. After the Degasperis-Procesi equation was derived, many papers were devoted to its study, cf. [8, 30, 33, 34, 55, 58, 59, 61, 73, 74, 75, 76]. When $s = k \geq 2$, the equation (1.2) becomes higher-order b-equation. In [62], Mu et. al studied the local well-posedness and global solutions for (1.2) (under a scaling transformation) with $k = 2$ in Sobolev spaces. In [9], Coclite et.al considered the cases $a = 2, k \geq 2, m = (1 - \partial_x^2 + \partial_x^4 - \dots + (-1)^k \partial_x^{2k})u$ — the higher-order Camassa-Holm equations, which describe the exponential curves of the manifold of smooth orientation-preserving diffeomorphisms of the unit circle in the plane. They established the existence of the unique global weak solutions.

For $a = 2$ and $\alpha = 0$, the system (1.1) becomes the two-component Camassa-Holm equation. Several types of 2-component Camassa-Holm equations have been studied in [7, 20, 28, 31, 32, 41, 42, 43, 44, 45, 46]. These works established the local well-posedness [20, 32, 41, 42], derived precise blow-up scenarios [32, 41], and proved that there exist strong solutions which blow up in finite time [20, 32, 42]. It also has global strong solutions [20, 42]. Moreover, it has global weak solutions [43, 44, 45, 46, 68, 69].

The system (1.1) with $s > 1$ was recently introduced by Escher and Lyons in [35]. It is the generalization of the same model (1.1) with $s = 1$ in [29]. In [29], the authors proved the local well-posedness of (1.1) with $s = 1$ by using a geometrical framework and they studied the blow-up scenarios

and global strong solutions of (1.1) in the periodic case. In [40], Guan et al. studied the local well-posedness of (1.1) with $s = 1$ on the line in supercritical Besov spaces, and several blow-up results and the persistence properties. In [52], He and Yin studied the local well-posedness of (1.1) with $s = 1$ in the critical Besov spaces on the line and the existence of analytic solutions of the system. In [35], for $s > 1$, by a geometric approach, the authors gave a blow-up criteria to ensure the geodesic completeness on the circle with $s > \frac{3}{2}, a = 2, \kappa \geq 0$ for the C^∞ initial data. In [53], the authors proved the local well-posedness results in Besov spaces with certain regularity condition, and gave some global existence results.

In this paper, we focus on the analyticity and Gevrey regularity of the system (1.1). Many researchers studied the analyticity for solutions to Camassa-Holm type systems, cf. [4, 50] and [71]. Recently, Luo and Yin [60] studied the Gevrey regularity of solutions to the Camassa-Holm type system by a generalized Ovsyannikov theorem (see Lemma 2.2 below). Applying this lemma, and following the ideas of [60], we obtain the local analyticity and Gevrey regularity of the solutions to system (1.1). Also, we prove the continuity of the data-to-solution map. Considering the existence of the global strong solution of system (1.1), with the idea from Levermore and Oliver [56] or Foias and Temam [39], we also study the global analyticity and Gevrey regularity of this system. First, we will show that, its solution is of analyticity and Gevrey regularity at least for a short time. Then, we will also show that, under certain conditions, the solution will keep in analyticity or Gevrey regularity for all $t \geq 0$.

Our paper is organized as follows. In Section 2, we give some preliminaries which will be used in Section 3. In Section 3, we establish the local analyticity and Gevrey regularity of the Cauchy problem associated with (1.1) and with (1.2). In Section 4, we discuss global analyticity and Gevrey regularity.

2 Preliminaries

We consider the Cauchy problem for the above system which can be rewritten in the following abstract form:

$$\begin{cases} \frac{du}{dt} = F(t, u(t)), \\ u|_{t=0} = u_0. \end{cases} \quad (2.1)$$

Lemma 2.1. [3, 64] *Let $\{X_\delta\}_{0 < \delta < 1}$ be a scale of decreasing Banach spaces, namely, for any $\delta' < \delta$ we have $X_\delta \subset X_{\delta'}$ and $\|\cdot\|_{\delta'} \leq \|\cdot\|_\delta$, and let $T, R > 0$. For given $u_0 \in X_1$, assume that:*

- (1) *If for $0 < \delta' < \delta < 1$ the function $t \mapsto u(t)$ is holomorphic in $|t| < T$ and continuous on $|t| \leq T$ with values in X_δ and*

$$\sup_{|t| < T} \|u(t)\|_\delta < R,$$

then $t \mapsto F(t, u(t))$ is a holomorphic function on $|t| < T$ with values in $X_{\delta'}$.

- (2) *For any $0 < \delta' < \delta < 1$ and any $u, v \in \overline{B(u_0, R)} \subset X_\delta$, there exists a positive constant L depending on u_0 and R such that*

$$\sup_{|t| < T} \|F(t, u) - F(t, v)\|_{\delta'} \leq \frac{L}{\delta - \delta'} \|u - v\|_\delta.$$

- (3) *For any $0 < \delta < 1$, there exists a positive constant M depending on u_0 such that*

$$\sup_{|t| < T} \|F(t, u_0)\|_\delta \leq \frac{M}{1 - \delta}.$$

Then there exist a $T_0 \in (0, T)$ and a unique solution to the Cauchy problem (2.1), which for every $\delta \in (0, 1)$ is holomorphic in $|t| < T_0(1 - \delta)$ with values in X_δ .

This theorem was first proposed by Ovsyannikov in [64, 65, 66]. However, the original Ovsyannikov theorem becomes invalid for the Gervy class. Because this kind of spaces do not satisfy the condition (2) of the Ovsyannikov theorem. More precisely, for the Gevrey class, we see that

$$\sup_{|t| < T} \|F(t, u) - F(t, v)\|_{\delta'} \leq \frac{L}{(\delta - \delta')^\sigma} \|u - v\|_\delta, \quad (2.2)$$

with $\sigma \geq 1$. If $\sigma > 1$, the above inequality is weaker than the condition (2) because it is of nonlinear decay. In [60], Luo and Yin established a new auxiliary function to obtain a generalized Ovsyannikov theorem by modifying the proof of [4].

Lemma 2.2. [60] *Let $\{X_\delta\}_{0 < \delta < 1}$ be a scale of decreasing Banach spaces, namely, for any $\delta' < \delta$ we have $X_\delta \subset X_{\delta'}$ and $\|\cdot\|_{\delta'} \leq \|\cdot\|_\delta$. Consider the Cauchy problem*

$$\begin{cases} \frac{du}{dt} = F(t, u(t)), \\ u|_{t=0} = u_0. \end{cases} \quad (2.3)$$

Let $T, R > 0, \sigma \geq 1$. For given $u_0 \in X_1$, assume that F satisfies the following three conditions:

- (1) *If for $0 < \delta' < \delta < 1$ the function $t \mapsto u(t)$ is holomorphic in $|t| < T$ and continuous on $|t| < T$ with values in X_δ and*

$$\sup_{|t| < T} \|u(t)\|_\delta < R,$$

then $t \mapsto F(t, u(t))$ is a holomorphic function on $|t| < T$ with values in $X_{\delta'}$.

- (2) *For any $0 < \delta' < \delta < 1$ and any $u, v \in \overline{B(u_0, R)} \subset X_\delta$, there exists a positive constant L depending on u_0 and R such that*

$$\sup_{|t| < T} \|F(t, u) - F(t, v)\|_{\delta'} \leq \frac{L}{(\delta - \delta')^\sigma} \|u - v\|_\delta.$$

- (3) *For any $0 < \delta < 1$, there exists a positive constant M depending u_0 such that*

$$\sup_{|t| < T} \|F(t, u_0)\|_\delta \leq \frac{M}{(1 - \delta)^\sigma}.$$

Then there exists a $T_0 \in (0, T)$ and a unique solution $u(t)$ to the Cauchy problem (2.3), which for every $\delta \in (0, 1)$ is holomorphic in $|t| < \frac{T_0(1-\delta)^\sigma}{2^{\sigma-1}}$ with values in X_δ .

Remark 2.3. In fact, $T_0 = \min\{\frac{1}{2^{2\sigma+4}L}, \frac{(2^\sigma-1)R}{(2^\sigma-1)2^{2\sigma+3}LR+M}\}$, which gives a lower bound of the lifespan.

Remark 2.4. The upper-bound condition “ $\delta < 1$ ” is not essential. Indeed, we can replace 1 by any other positive δ_0 and obtain the similar result.

Remark 2.5. If $\sigma = 1$ Lemma 2.2 reduces to the so called abstract Cauchy-Kovalevsky theorem.

To apply Lemma 2.2, we first introduce the Sobolev-Gevrey spaces and recall some basic properties.

Definition 2.6. [39] Let s be a real number and $\sigma, \delta > 0$. A function $f \in G_{\sigma,s}^\delta(\mathbb{R}^d)$ if and only if $f \in C^\infty(\mathbb{R}^d)$ and satisfies

$$\|f\|_{G_{\sigma,s}^\delta(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} (1 + |\xi|^2)^s e^{2\delta(1+|\xi|^2)^{\frac{1}{2\sigma}}} |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}} < \infty.$$

A function $f \in \bar{G}_{\sigma,s}^\delta(\mathbb{R}^d)$ if and only if $f \in C^\infty(\mathbb{R}^d)$ and satisfies

$$\|f\|_{\bar{G}_{\sigma,s}^\delta(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} (1 + |\xi|^2)^s e^{2\delta|\xi|^{\frac{1}{\sigma}}} |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}} < \infty.$$

Definition 2.7. [57] A function is of Gevrey class $\sigma \geq 1$, if there exist $\delta > 0, r \geq 0$ such that $f \in G_{\sigma,r}^\delta(\mathbb{R}^d)$. Denote the the functions of Gevrey class σ

$$f \in G_\sigma(\mathbb{R}^d) := \bigcup_{\delta > 0, r \in \mathbb{R}} G_{\sigma,r}^\delta(\mathbb{R}^d).$$

Remark 2.8. Denoting the Fourier multiplier $e^{\delta A^{\frac{1}{\sigma}}}$ and $e^{\delta(-\delta)^{\frac{1}{2\sigma}}}$ by

$$e^{\delta A^{\frac{1}{\sigma}}} f = \mathcal{F}^{-1}(e^{\delta(1+|\xi|^2)^{\frac{1}{2\sigma}}} \hat{f}) \quad \text{and} \quad e^{\delta(-\delta)^{\frac{1}{2\sigma}}} f = \mathcal{F}^{-1}(e^{\delta|\xi|^{\frac{1}{\sigma}}} \hat{f})$$

respectively, we deduce that $\|f\|_{G_{\sigma,s}^\delta(\mathbb{R}^d)} = \|e^{\delta A^{\frac{1}{\sigma}}} f\|_{H^s(\mathbb{R}^d)}$ and $\|f\|_{\bar{G}_{\sigma,s}^\delta(\mathbb{R}^d)} = \|e^{\delta(-\delta)^{\frac{1}{2\sigma}}} f\|_{H^s(\mathbb{R}^d)}$. Note that for $\sigma \geq 1$,

$$|\xi|^{\frac{1}{\sigma}} \leq (1 + |\xi|^2)^{\frac{1}{2\sigma}} \leq 1 + |\xi|^{\frac{1}{\sigma}}.$$

It follows that

$$\|f\|_{\bar{G}_{\sigma,s}^\delta} \leq \|f\|_{G_{\sigma,s}^\delta} \leq e^\delta \|f\|_{\bar{G}_{\sigma,s}^\delta}.$$

For $0 < \sigma < 1$, it is called ultra-analytic function. If $\sigma = 1$, it is usual analytic function and δ is called the radius of analyticity. If $\sigma > 1$, it is the Gevrey class function.

Proposition 2.9. Let $0 < \delta' < \delta$, $0 < \sigma' < \sigma$ and $s' < s$. From Definition 2.6, one can check that $G_{\sigma,s}^\delta(\mathbb{R}^d) \hookrightarrow G_{\sigma',s}^{\delta'}(\mathbb{R}^d)$, $G_{\sigma',s}^{\delta'}(\mathbb{R}^d) \hookrightarrow G_{\sigma,s}^\delta(\mathbb{R}^d)$ and $G_{\sigma,s}^\delta(\mathbb{R}^d) \hookrightarrow G_{\sigma,s'}^\delta(\mathbb{R}^d)$, with the embedding constants $C = 1$.

Along the similar computations of the proof of Proposition 2.4–2.5 in [60], we can obtain the following two propositions.

Proposition 2.10. Let s be a real number and $\sigma > 0$. Assume that $0 < \delta' < \delta$. Then we have

$$\|\partial_x f\|_{G_{\sigma,s}^{\delta'}(\mathbb{R})} \leq \frac{e^{-\sigma} \sigma^\sigma}{(\delta - \delta')^\sigma} \|f\|_{G_{\sigma,s}^\delta(\mathbb{R})}.$$

Proposition 2.11. Let $s > \frac{1}{2}$, $\sigma \geq 1$ and $\delta > 0$. Then $G_{\sigma,s}^\delta(\mathbb{R})$ is an algebra. Moreover, there exists a constant C_s such that

$$\|fg\|_{G_{\sigma,s}^\delta(\mathbb{R})} \leq C_s \|f\|_{G_{\sigma,s}^\delta(\mathbb{R})} \|g\|_{G_{\sigma,s}^\delta(\mathbb{R})}.$$

Now we can state some known results of the system (1.1), which will be used in sequel.

Lemma 2.12. [53] Let $a = 2$, $s = [s] \geq 2$, $q > s + \frac{1}{2}$, then the solution to (1.2) with the initial data $u_0 \in H^q$ exists globally in time.

Lemma 2.13. [53] Suppose $s > \frac{3}{2}$. If $a = 2$ and $q \geq 2s$, then the solution to (1.2) with the initial data $u_0 \in H^q$ exists globally in time.

Lemma 2.14. [53] Suppose $s = [s] = k \geq 2$, $a = 2$, $\kappa \geq 0$, $q > s + \frac{1}{2}$ and q_1 satisfies the following condition

$$\frac{1}{2} < q_1 \leq q - 1 \leq q_1 + 2s - 2. \quad (2.4)$$

Given the initial data $(u_0, \rho_0) \in H^q \times H^{q_1}$, then the solution (u, ρ) to (1.1) exists globally in time, namely, $(u, \rho) \in C([0, \infty); H^q \times H^{q_1})$.

Lemma 2.15. [53] Suppose $a = 2$, $\kappa \geq 0$, $s > \frac{3}{2}$, $q \geq 2s$ and q_1 satisfies the condition (2.4). Given the initial data $(u_0, \rho_0) \in H^q \times H^{q_1}$, then the solution (u, ρ) to (1.1) exists globally in time, namely, $(u, \rho) \in C([0, \infty); H^q \times H^{q_1})$.

For simplicity, we will only consider the integer case that $s = [s] \geq 2$ in the following part of our paper.

3 Local Gevrey regularity and analyticity

Now we can present a main theorem of our paper.

Theorem 3.1. *Let $\sigma \geq 1$ and $q > s + \frac{1}{2}$. Assume that $u_0 \in G_{\sigma,q}^1(\mathbb{R})$. Then for every $0 < \delta < 1$, there exists a T_0 such that (1.2) has a unique solution u which is holomorphic in $|t| < \frac{T_0(1-\delta)^\sigma}{2^\sigma-1}$ with values in $G_{\sigma,q}^\delta(\mathbb{R})$. Moreover, $T_0 \approx \frac{1}{\|u_0\|_{G_{\sigma,q}^1(\mathbb{R})}}$.*

The one-component equation (1.2) can be rewritten as (see [53])

$$\begin{aligned} u_t + uu_x &= K(u, u) \\ &\sim (1 - \partial_x^2)^{-s} \left(\partial_x^{2s-1}(u_x^2) + \partial_x^{2s-3}(u_{xx}^2) + \dots \right. \\ &\quad \left. + \partial_x[(\partial_x^s u)^2] + \partial_x^{2s-3}(u_x^2) + \dots + \partial_x[(\partial_x^{s-1} u)^2] + \dots + \partial_x(u^2) \right). \end{aligned} \quad (3.1)$$

In order to use Lemma 2.2, we rewrite it as

$$\begin{cases} u_t = F(u) := -u\partial_x u + K(u, u), \\ u|_{t=0} = u_0. \end{cases}$$

For a fixed $\sigma \geq 1$ and $q > s + \frac{1}{2}$, Proposition 2.9 ensures that $\{G_{\sigma,q}^\delta\}_{0 < \delta < 1}$ is a scale of decreasing Banach spaces. For any $0 < \delta' < \delta$, we need to estimate

$$\|F(u)\|_{G_{\sigma,q}^{\delta'}} \leq \frac{1}{2} \|\partial_x(u^2)\|_{G_{\sigma,q}^{\delta'}} + \|K(u, u)\|_{G_{\sigma,q}^{\delta'}}.$$

Note that $\frac{1}{2} \|\partial_x(u^2)\|_{G_{\sigma,q}^{\delta'}} \leq \frac{e^{-\sigma}\sigma^\sigma}{2(\delta-\delta')^\sigma} \|u^2\|_{G_{\sigma,q}^\delta} \leq C_q \frac{e^{-\sigma}\sigma^\sigma}{2(\delta-\delta')^\sigma} \|u\|_{G_{\sigma,q}^\delta}^2$. For any integers $1 \leq i \leq s, 0 \leq j \leq s$ such that $(2i-1) + 2j \leq 2s+1$, we have

$$\|(1 - \partial_x^2)^{-s} \partial_x^{2i-1}[(\partial_x^j u)^2]\|_{G_{\sigma,q}^{\delta'}} \leq C_q \|(\partial_x^j u)^2\|_{G_{\sigma,q-2s+2i-1}^{\delta'}} \leq C_q \|(\partial_x^j u)^2\|_{G_{\sigma,q-j}^{\delta'}} \leq C_q \|u\|_{G_{\sigma,q}^\delta}^2.$$

Hence

$$\|K(u, u)\|_{G_{\sigma,q}^{\delta'}} \leq C_{q,s} \|u\|_{G_{\sigma,q}^\delta}^2.$$

Thus, in view of $(\delta - \delta')^\sigma < 1$, we have

$$\|F(u)\|_{G_{\sigma,q}^{\delta'}} \leq \frac{C_{q,s}(e^{-\sigma}\sigma^\sigma + 2)}{2(\delta - \delta')^\sigma} \|u\|_{G_{\sigma,q}^\delta}^2,$$

which implies that F satisfies the condition (1) of Lemma 2.2. The similar computations yields

$$\|F(u_0)\|_{G_{\sigma,q}^\delta} \leq \frac{C_{q,s}(e^{-\sigma}\sigma^\sigma + 2)}{2(1-\delta)^\sigma} \|u_0\|_{G_{\sigma,q}^1}^2,$$

which implies that F satisfies the condition (3) of Lemma 2.2 with $M = C_{q,s}(\frac{e^{-\sigma}\sigma^\sigma}{2} + 1) \|u_0\|_{G_{\sigma,q}^1}^2$. It remains to verify that F satisfies the condition (2) of Theorem 2.2. Assume that $\|u - u_0\|_{G_{\sigma,q}^\delta} \leq R$ and $\|v - v_0\|_{G_{\sigma,q}^\delta} \leq R$. Applying Propositions 2.10, we get

$$\begin{aligned} \|F(u) - F(v)\|_{G_{\sigma,q}^{\delta'}} &= \left\| -\frac{1}{2} \partial_x(u^2 - v^2) + (K(u, u) - K(v, v)) \right\|_{G_{\sigma,q}^{\delta'}} \\ &\leq \frac{e^{-\sigma}\sigma^\sigma}{2(\delta - \delta')^\sigma} \|(u+v)(u-v)\|_{G_{\sigma,q}^\delta} + \|K(u, u) - K(v, v)\|_{G_{\sigma,q}^{\delta'}}, \end{aligned}$$

where

$$K(u, u) - K(v, v) \sim (1 - \partial_x^2)^{-s} \left(\partial_x^{2s-1}(u_x^2 - v_x^2) + \partial_x^{2s-3}(u_{xx}^2 - v_{xx}^2) + \dots + \partial_x[(\partial_x^s u)^2 - (\partial_x^s v)^2] + \right. \\ \left. + \partial_x^{2s-3}(u_x^2 - v_x^2) + \dots + \partial_x[(\partial_x^{s-1} u - (\partial_x^{s-1} v)^2] + \dots + \partial_x(u^2 - v^2) \right).$$

Note that

$$\|(u+v)(u-v)\|_{G_{\sigma,q}^\delta} \leq C_q \|u+v\|_{G_{\sigma,q}^\delta} \|u-v\|_{G_{\sigma,q}^\delta} \leq C_q (2\|u_0\|_{G_{\sigma,q}^\delta} + 2R) \|u-v\|_{G_{\sigma,q}^\delta}.$$

On the other hand, for any integers $1 \leq i \leq s, 0 \leq j \leq s$ such that $(2i-1) + 2j \leq 2s+1$, we obtain

$$\begin{aligned} \|(1 - \partial_x^2)^{-s} \partial_x^{2i-1}[(\partial_x^j u)^2 - (\partial_x^j v)^2]\|_{G_{\sigma,q}^{\delta'}} &\leq C_q \|(\partial_x^j u)^2 - (\partial_x^j v)^2\|_{G_{\sigma,q-2s+2i-1}^\delta} \\ &\leq C_q \|\partial_x^j(u+v)\partial_x^j(u-v)\|_{G_{\sigma,q-j}^\delta} \\ &\leq C_q \|u+v\|_{G_{\sigma,q}^\delta} \|u-v\|_{G_{\sigma,q}^\delta} \\ &\leq C_q (2\|u_0\|_{G_{\sigma,q}^\delta} + 2R) \|u-v\|_{G_{\sigma,q}^\delta}. \end{aligned}$$

Therefore, we have $\|K(u, u) - K(v, v)\|_{G_{\sigma,q}^{\delta'}} \leq C_{q,s} (2\|u_0\|_{G_{\sigma,q}^\delta} + 2R) \|u-v\|_{G_{\sigma,q}^\delta}$, and

$$\|F(u) - F(v)\|_{G_{\sigma,q}^{\delta'}} \leq \frac{C_{q,s}(e^{-\sigma}\sigma^\sigma + 2)}{(\delta - \delta')^\sigma} (\|u_0\|_{G_{\sigma,q}^1} + R) \|u-v\|_{G_{\sigma,q}^\delta}.$$

Thus F satisfies the condition (2) of Lemma 2.2 with $L = C_{q,s}(e^{-\sigma}\sigma^\sigma + 2)(\|u_0\|_{G_{\sigma,q}^1} + R)$. Hence we obtain the local existence result of (1.2) with the Gevrey regularity or analyticity, and $T_0 = \min\{\frac{1}{2^{2\sigma+4}L}, \frac{(2^\sigma-1)R}{(2^\sigma-1)2^{2\sigma+3}LR+M}\}$. By setting $R = \|u_0\|_{G_{\sigma,q}^1}$, we see that $L = 2C_{q,s}(e^{-\sigma}\sigma^\sigma + 2)\|u_0\|_{G_{\sigma,q}^1}$ and $M \leq 2^{2\sigma+3}LR$, and hence

$$T_0 = \min\left\{\frac{1}{2^{2\sigma+4}L}, \frac{(2^\sigma-1)R}{(2^\sigma-1)2^{2\sigma+3}LR+M}\right\} = \frac{1}{2^{2\sigma+5}C_{q,s}(e^{-\sigma}\sigma^\sigma + 2)\|u_0\|_{G_{\sigma,q}^1}}. \quad \square$$

We state another theorem to present the local Gevrey regularity and analyticity for the two-component system (1.1).

Theorem 3.2. *Let $\sigma \geq 1$ and $q > s + \frac{1}{2}$. Assume that $u_0 \in G_{\sigma,q}^1(\mathbb{R})$ and $\rho_0 \in G_{\sigma,q_1}^1(\mathbb{R})$. Then for every $0 < \delta < 1$, there exists a $T_0 > 0$ such that the two-component system (1.1) has a unique solution (u, ρ) which is holomorphic in $|t| < \frac{T_0(1-\delta)^\sigma}{2^\sigma-1}$ with values in $G_{\sigma,q}^\delta(\mathbb{R}) \times G_{\sigma,q_1}^\delta(\mathbb{R})$. Moreover, we can have*

$$T_0 \approx \frac{1}{\|u_0\|_{G_{\sigma,q}^1} + \|\rho_0\|_{G_{\sigma,q_1}^1} + 1}.$$

Proof. We rewrite the two-component system (1.1) into the following form

$$\begin{cases} z_t = F(z), \\ z|_{t=0} = z_0, \end{cases}$$

with $z = (u, \rho)^T, z_0 = (u_0, \rho_0)^T$ and

$$F(z) = \begin{pmatrix} F_1(z) \\ F_2(z) \end{pmatrix} := \begin{pmatrix} -\frac{1}{2}\partial_x(u^2) + K(u, u) + (1 - \partial_x^2)^{-s}(\alpha u_x - \frac{\kappa}{2}\partial_x(\rho^2)) \\ -\partial_x(u\rho) + (2-a)u_x\rho \end{pmatrix}. \quad (3.2)$$

For fixed $\sigma \geq 1$ and $s > \frac{3}{2}$, we set $X_\delta = G_{\sigma,q}^\delta(\mathbb{R}) \times G_{\sigma,q_1}^\delta(\mathbb{R})$ and

$$\|z\|_\delta = \|u\|_{G_{\sigma,q}^\delta} + \|\rho\|_{G_{\sigma,q_1}^\delta}.$$

Proposition 2.9 then ensures that $\{X_\delta\}_{0 < \delta < 1}$ is a scale of decreasing Banach spaces. From the proof of Theorem 3.1, we have shown that for any $0 < \delta' < \delta$,

$$\| -\frac{1}{2}\partial_x(u^2) + K(u, u) \|_{G_{\sigma,q}^{\delta'}} \leq \frac{C_{q,s}(e^{-\sigma}\sigma^\sigma + 2)}{2(\delta - \delta')^\sigma} \|u\|_{G_{\sigma,q}^\delta}^2.$$

Note that $\|(1 - \partial_x^2)^{-s}u_x\|_{G_{\sigma,q}^{\delta'}} \leq \|u\|_{G_{\sigma,q-2s+1}^\delta} \leq \|u\|_{G_{\sigma,q}^\delta}$ and

$$\|(1 - \partial_x^2)^{-s}\partial_x(\rho^2)\|_{G_{\sigma,q}^{\delta'}} \leq \|\rho^2\|_{G_{\sigma,q-2s+1}^\delta} \leq \|\rho^2\|_{G_{\sigma,q_1}^\delta} \leq C_{q_1}\|\rho\|_{G_{\sigma,q_1}^\delta}^2.$$

Hence, we obtain

$$\|F_1(z)\|_{G_{\sigma,q}^{\delta'}} \leq \frac{C_{q,s}(e^{-\sigma}\sigma^\sigma + 2)}{2(\delta - \delta')^\sigma} \|u\|_{G_{\sigma,q}^\delta}^2 + |\alpha| \cdot \|u\|_{G_{\sigma,q}^\delta} + \frac{|\kappa|}{2} C_{q_1} \|\rho\|_{G_{\sigma,q_1}^\delta}^2.$$

On the other hand, considering the second component, we see

$$\begin{aligned} \|F_2(z)\|_{G_{\sigma,q_1}^{\delta'}} &= \| -\partial_x(u\rho) + (2-a)u_x\rho \|_{G_{\sigma,q_1}^{\delta'}} \\ &\leq \frac{e^{-\sigma}\sigma^\sigma}{(\delta - \delta')^\sigma} \|u\rho\|_{G_{\sigma,q_1}^\delta} + |2-a| \cdot \|u_x\rho\|_{G_{\sigma,q_1}^{\delta'}} \\ &\leq C_{q_1} \frac{e^{-\sigma}\sigma^\sigma}{(\delta - \delta')^\sigma} \|u\|_{G_{\sigma,q}^\delta} \|\rho\|_{G_{\sigma,q_1}^\delta} + |2-a| \cdot C_{q_1} \cdot \|u_x\|_{G_{\sigma,q}^{\delta'}} \|\rho\|_{G_{\sigma,q_1}^{\delta'}} \\ &\leq \frac{(|2-a|+1)C_{q_1}e^{-\sigma}\sigma^\sigma}{(\delta - \delta')^\sigma} \|u\|_{G_{\sigma,q}^\delta} \|\rho\|_{G_{\sigma,q_1}^\delta}. \end{aligned}$$

Then we can obtain

$$\|F(z)\|_{\delta'} = \|F_1(z)\|_{G_{\sigma,q}^{\delta'}} + \|F_2(z)\|_{G_{\sigma,q_1}^{\delta'}} \leq C_{q,q_1,a,s,\alpha,\kappa} \frac{e^{-\sigma}\sigma^\sigma + 2}{(\delta - \delta')^\sigma} (\|u\|_{G_{\sigma,q}^\delta} + \|\rho\|_{G_{\sigma,q_1}^\delta} + 1)^2,$$

and F satisfies the condition (1) of Lemma 2.2. By the similar computations, we obtain that

$$\|F(z_0)\|_\delta \leq C_{q,q_1,a,s,\alpha,\kappa} (e^{-\sigma}\sigma^\sigma + 2) (\|u_0\|_{G_{\sigma,q}^1} + \|\rho_0\|_{G_{\sigma,q_1}^1} + 1)^2 \frac{1}{(1 - \delta)^\sigma},$$

which means that F satisfies the condition (3) of the Lemma 2.2 with

$$M = C_{q,q_1,a,s,\alpha,\kappa} (e^{-\sigma}\sigma^\sigma + 2) (\|z_0\|_1 + 1)^2.$$

It remains to show that F satisfies the condition (2) of Lemma 2.2. Assume that

$$\|z_1 - z_0\|_\delta \leq R, \quad \|z_2 - z_0\|_\delta \leq R,$$

one can obtain

$$\begin{aligned} &\|F_1(z_1) - F_1(z_2)\|_{G_{\sigma,q}^{\delta'}} \\ &\leq \| -\frac{1}{2}\partial_x(u_1^2 - u_2^2) + (K(u_1, u_1) - K(u_2, u_2)) \|_{G_{\sigma,q}^{\delta'}} + \|(1 - \partial_x^2)^{-s}(\alpha(u_1 - u_2)_x - \frac{\kappa}{2}(\rho_1^2 - \rho_2^2)_x)\|_{G_{\sigma,q}^{\delta'}} \\ &\leq \frac{C_{q,s}(e^{-\sigma}\sigma^\sigma + 2)}{(\delta - \delta')^\sigma} (\|u_0\|_{G_{\sigma,q}^1} + R) \|u_1 - u_2\|_{G_{\sigma,q}^\delta} \\ &\quad + |\alpha| \|u_1 - u_2\|_{G_{\sigma,q-2s+1}^\delta} + \frac{|\kappa|}{2} \|(\rho_1 + \rho_2)(\rho_1 - \rho_2)\|_{G_{\sigma,q-2s+1}^\delta}, \end{aligned}$$

with

$$\begin{aligned}
 \|(\rho_1 + \rho_2)(\rho_1 - \rho_2)\|_{G_{\sigma, q-2s+1}^\delta} &\leq \|(\rho_1 + \rho_2)(\rho_1 - \rho_2)\|_{G_{\sigma, q_1}^\delta} \\
 &\leq C_{q_1} (\|\rho_1\|_{G_{\sigma, q_1}^\delta} \|\rho_2\|_{G_{\sigma, q_1}^\delta}) \|\rho_1 - \rho_2\|_{G_{\sigma, q_1}^\delta} \\
 &\leq C_{q_1} (2\|\rho_0\|_{G_{\sigma, q_1}^\delta} + 2R) \|\rho_1 - \rho_2\|_{G_{\sigma, q_1}^\delta}.
 \end{aligned}$$

It then follows that

$$\|F_1(z_1) - F_1(z_2)\|_{G_{\sigma, q}^{\delta'}} \leq \frac{C_{q, q_1, a, s, \alpha, \kappa} (e^{-\sigma} \sigma^\sigma + 2) (\|z_0\|_1 + 1 + R)}{(\delta - \delta')^\sigma} \|z_1 - z_2\|_\delta.$$

On the other hand, we see

$$\begin{aligned}
 &\|F_2(z_1) - F_2(z_2)\|_{G_{\sigma, q_1}^{\delta'}} \\
 &= \|\partial_x(u_1 \rho_1 - u_2 \rho_2) + (2 - a)(u_{1x} \rho_1 - u_{2x} \rho_2)\|_{G_{\sigma, q_1}^{\delta'}} \\
 &\leq \frac{e^{-\sigma} \sigma^\sigma}{(\delta - \delta')^\sigma} \|u_1(\rho_1 - \rho_2) + (u_1 - u_2)\rho_2\|_{G_{\sigma, q_1}^\delta} + |2 - a| \cdot \|(u_1 - u_2)_x \rho_1 + u_{2x}(\rho_1 - \rho_2)\|_{G_{\sigma, q_1}^{\delta'}}.
 \end{aligned}$$

Since q, q_1 satisfies the condition (2.4), according to Lemma 2.11, we have

$$\begin{aligned}
 \|u_1(\rho_1 - \rho_2) + (u_1 - u_2)\rho_2\|_{G_{\sigma, q_1}^\delta} &\leq C_{q_1} \|u_1\|_{G_{\sigma, q}^\delta} \|\rho_1 - \rho_2\|_{G_{\sigma, q_1}^\delta} + C_{q_1} \|u_1 - u_2\|_{G_{\sigma, q}^\delta} \|\rho_2\|_{G_{\sigma, q_1}^\delta} \\
 &\leq C_{q_1} (\|z_0\|_\delta + R) \|z_1 - z_2\|_\delta,
 \end{aligned}$$

and

$$\begin{aligned}
 &\|(u_1 - u_2)_x \rho_1 + u_{2x}(\rho_1 - \rho_2)\|_{G_{\sigma, q_1}^{\delta'}} \\
 &\leq C_{q_1} (\|(u_1 - u_2)_x\|_{G_{\sigma, q_1}^{\delta'}} \|\rho_1\|_{G_{\sigma, q_1}^{\delta'}} + \|u_{2x}\|_{G_{\sigma, q_1}^{\delta'}} \|\rho_1 - \rho_2\|_{G_{\sigma, q_1}^{\delta'}}) \\
 &\leq C_{q_1} \frac{e^{-\sigma} \sigma^\sigma}{(\delta - \delta')^\sigma} \left\{ \|u_1 - u_2\|_{G_{\sigma, q}^\delta} (\|\rho_0\|_{G_{\sigma, q_1}^\delta} + R) + (\|u_0\|_{G_{\sigma, q}^\delta} + R) \|\rho_1 - \rho_2\|_{G_{\sigma, q_1}^\delta} \right\} \\
 &\leq C_{q_1} \frac{e^{-\sigma} \sigma^\sigma}{(\delta - \delta')^\sigma} (\|z_0\|_\delta + R) \|z_1 - z_2\|_\delta.
 \end{aligned}$$

and hence

$$\|F_2(z_1) - F_2(z_2)\|_{G_{\sigma, q_1}^{\delta'}} \leq C_{q_1} (1 + |a - 2|) \frac{e^{-\sigma} \sigma^\sigma}{(\delta - \delta')^\sigma} (\|z_0\|_1 + R) \|z_1 - z_2\|_\delta.$$

Therefore, we deduce that

$$\begin{aligned}
 \|F(z_1) - F(z_2)\|_{\delta'} &\leq \|F_1(z_1) - F_1(z_2)\|_{G_{\sigma, q}^{\delta'}} + \|F_2(z_1) - F_2(z_2)\|_{G_{\sigma, q_1}^{\delta'}} \\
 &\leq \frac{C_{q, q_1, a, s, \alpha, \kappa} (e^{-\sigma} \sigma^\sigma + 2) (\|z_0\|_1 + 1 + R)}{(\delta - \delta')^\sigma} \|z_1 - z_2\|_\delta,
 \end{aligned} \tag{3.3}$$

and F satisfies the condition (2) of Lemma 2.2 with $L = C_{q, q_1, a, s, \alpha, \kappa} (e^{-\sigma} \sigma^\sigma + 2) (\|z_0\|_1 + 1 + R)$. Hence we obtain the local existence result of (1.2) with the Gevrey regularity or analyticity, and

$$T_0 = \min\left\{\frac{1}{2^{2\sigma+4}L}, \frac{(2^\sigma - 1)R}{(2^\sigma - 1)2^{2\sigma+3}LR + M}\right\}.$$

Moreover, by setting $R = \|z_0\|_1 + 1$, we can see $L = 2C_{q, q_1, a, s, \alpha, \kappa} (e^{-\sigma} \sigma^\sigma + 2) (\|z_0\|_1 + 1)$ and $M \leq 2^{2\sigma+3}LR$. It then follows that

$$T_0 = \frac{1}{2^{2\sigma+5}C_{q, q_1, a, s, \alpha, \kappa}(e^{-\sigma} \sigma^\sigma + 2)(\|z_0\|_1 + 1)}.$$

This completes the proof of Theorem 3.2. \square

4 Continuity of the data-to-solution map

In this section, we study the continuity of the data-to-solution map for initial data and solutions in Theorems 3.1–3.2. We only prove this for the two-component system (1.1).

At first we introduce a definition to explain what means the data-to-solution map is continuous from $G_{\sigma,q}^1(\mathbb{R}) \times G_{\sigma,q_1}^1(\mathbb{R})$ into the solution space.

Definition 4.1. Let $\sigma \geq 1, q > s + \frac{1}{2}$ and let q_1 satisfy the condition (2.4). We say that the data-to-solution map $(u_0, \rho_0) \mapsto (u, \rho)$ of the system (1.1) is continuous if for a given initial datum $(u_0^\infty, \rho_0^\infty) \in G_{\sigma,q}^1 \times G_{\sigma,q_1}^1$, there exists a $T = T(\|u_0^\infty\|_{G_{\sigma,q}^1}, \|\rho_0^\infty\|_{G_{\sigma,q_1}^1}) > 0$, such that for any sequence $(u_0^n, \rho_0^n) \in G_{\sigma,q}^1 \times G_{\sigma,q_1}^1$ and $\|u_0^n - u_0^\infty\|_{G_{\sigma,q}^1} + \|\rho_0^n - \rho_0^\infty\|_{G_{\sigma,q_1}^1} \xrightarrow{n \rightarrow \infty} 0$, the corresponding solutions (u^n, ρ^n) of system (1.1) satisfy $\|z^n - z^\infty\|_{E_T} := \|u^n - u^\infty\|_{E_{q,T}} + \|\rho^n - \rho^\infty\|_{E_{q_1,T}} \xrightarrow{n \rightarrow \infty} 0$, where

$$\|f\|_{E_{q,T}} := \sup_{|t| < \frac{T(1-\delta)^\sigma}{2^\sigma - 1}} \left(\|f(t)\|_{G_{\sigma,q}^\delta} (1-\delta)^\sigma \sqrt{1 - \frac{|t|}{T(1-\delta)^\sigma}} \right).$$

Also, we need to introduce the following lemma.

Lemma 4.2. [60] Let $\sigma \geq 1$. For every $a > 0, u \in E_a, 0 < \delta < 1$ and $0 \leq t < \frac{a(1-\delta)^\sigma}{2^\sigma - 1}$ we have

$$\int_0^t \frac{\|u(\tau)\|_{\delta(\tau)}}{(\delta(\tau) - \delta)^\sigma} d\tau \leq \frac{a2^{2\sigma+3}\|u\|_{E_a}}{(1-\delta)^\sigma} \sqrt{\frac{a(1-\delta)^\sigma}{a(1-\delta)^\sigma - t}},$$

where $\delta(t) = \frac{1}{2}(1+\delta) + (\frac{1}{2})^{2+\frac{1}{\delta}} \left\{ [(1-\delta)^\sigma - \frac{t}{a}]^{\frac{1}{\sigma}} - [(1-\delta)^\sigma + (2^{\sigma+1} - 1)\frac{t}{a}]^{\frac{1}{\sigma}} \right\} \in (\delta, 1)$.

Now we can state the main theorem of this section.

Theorem 4.3. Let $\sigma \geq 1, q > s + \frac{1}{2}$ and let q_1 satisfy the condition (2.4). Assume $(u_0, \rho) \in G_{\sigma,q}^1(\mathbb{R}) \times G_{\sigma,q_1}^1(\mathbb{R})$. Then the data-to-solution map $(u_0, \rho_0) \mapsto (u, \rho)$ of the system (1.1) is continuous from $G_{\sigma,q}^1 \times G_{\sigma,q_1}^1$ into the solution space.

Proof. Without loss of generality, we may assume that $t \geq 0$. As in the proof of Theorem 3.2, we use the notations $z^n = (u^n, \rho^n)^T, z_0^n = (u_0^n, \rho_0^n)^T, \|z^n\|_\delta = \|u^n\|_{G_{\sigma,q}^\delta} + \|\rho^n\|_{G_{\sigma,q_1}^\delta}$ and $\|z\|_{E_T} = \|u^n\|_{E_{q,T}} + \|\rho^n\|_{E_{q_1,T}}$. Define that

$$T^\infty = \frac{1}{2^{2\sigma+5}C_{q,q_1,s,\alpha,\kappa}(e^{-\sigma}\sigma^\sigma + 2)(\|z_0^\infty\|_1 + 1)}, \quad T^n = \frac{1}{2^{2\sigma+5}C_{q,q_1,s,\alpha,\kappa}(e^{-\sigma}\sigma^\sigma + 2)(\|z_0^n\|_1 + 1)},$$

where $C_{q,q_1,s,\alpha,\kappa}$ is given in (3.3). Since $\|z_0^n - z_0^\infty\|_1 \xrightarrow{n \rightarrow \infty} 0$, it follows that there exists a constant N such that if $n \geq N$, we can have

$$\|z_0^n\|_1 \leq \|z_0^\infty\|_1 + 1. \quad (4.1)$$

By setting

$$T = \frac{1}{2^{2\sigma+5}C_{q,q_1,s,\alpha,\kappa}(e^{-\sigma}\sigma^\sigma + 2)(\|z_0^n\|_1 + 2)}, \quad (4.2)$$

we deduce from (4.1) that $T < \min\{T^n, T^\infty\}$ for any $n \geq N$. As in the proof of Theorem 3.2, we see that T^n and T^∞ are the existence time of the solutions z^n and z^∞ corresponding to z_0^n and z_0^∞ respectively. Thus, we can see, for any $n \geq N$,

$$z^\infty(t, x) = z_0^\infty(x) + \int_0^t F(z^\infty(t, x)) d\tau, \quad 0 \leq t < \frac{T(1-\delta)^\sigma}{2^\sigma - 1},$$

$$z^n(t, x) = z_0^n(x) + \int_0^t F(z^n(\tau, x)) d\tau, \quad 0 \leq t < \frac{T(1-\delta)^\sigma}{2^\sigma - 1},$$

where F is given in (3.2). Therefore, for any $0 \leq t < \frac{T(1-\delta)^\sigma}{2^\sigma - 1}$ and $0 < \delta < 1$, we have

$$\|z^n(t) - z^\infty(t)\|_\delta \leq \|z_0^n - z_0^\infty\|_\delta + \int_0^t \|F(z^n(\tau)) - F(z^\infty(\tau))\|_\delta d\tau. \quad (4.3)$$

Define that $\delta(t) = \frac{1}{2}(1 + \delta) + (\frac{1}{2})^{2+\frac{1}{\delta}} \{[(1-\delta)^\sigma - \frac{t}{T}]^{\frac{1}{\sigma}} - [(1-\delta)^\sigma + (2^{\sigma+1} - 1)\frac{t}{T}]^{\frac{1}{\sigma}}\}$. By virtue of Lemma 4.2 with $a = T$, we see that $\delta < \delta(\tau) < 1$. Taking advantage of (3.3), we obtain

$$\|F(z^n(\tau)) - F(z^\infty(\tau))\|_\delta \leq L \frac{\|z^n(\tau) - z^\infty(\tau)\|_{\delta(\tau)}}{(\delta(\tau) - \delta)^\sigma} \quad (4.4)$$

with $L = 2C_{q,q_1,a,s,\alpha,\kappa} (e^{-\sigma} \sigma^\sigma + 2)(\|z_0\|_1 + 1)$. Plugging it into (4.3) yields that

$$\|z^n(t) - z^\infty(t)\|_\delta \leq \|z_0^n - z_0^\infty\|_\delta + L \int_0^t \frac{\|z^n(\tau) - z^\infty(\tau)\|_{\delta(\tau)}}{(\delta(\tau) - \delta)^\sigma} d\tau.$$

Applying Lemma 4.2, we deduce that

$$\|z^n(t) - z^\infty(t)\|_\delta \leq \|z_0^n - z_0^\infty\|_\delta + L \frac{T 2^{2\sigma+3} \|z^n - z^\infty\|_{E_T}}{(\delta(1-\delta)^\sigma)} \sqrt{\frac{T(1-\delta)^\sigma}{T(1-\delta)^\sigma - t}}.$$

Noting that $LT 2^{2\sigma+3} < \frac{1}{2}$, we can obtain

$$\|z^n(t) - z^\infty(t)\|_\delta \leq \|z_0^n - z_0^\infty\|_\delta + \frac{1}{2(1-\delta)^\delta} \|z^n - z^\infty\|_{E_T} \sqrt{\frac{T(1-\delta)^\sigma}{T(1-\delta)^\sigma - t}},$$

which leads to

$$\begin{aligned} \|z^n(t) - z^\infty(t)\|_\delta (1-\delta)^\sigma \sqrt{1 - \frac{t}{T(1-\delta)^\sigma}} &\leq \|z_0^n - z_0^\infty\|_\delta (1-\delta)^\sigma \sqrt{1 - \frac{t}{T(1-\delta)^\sigma}} + \frac{1}{2} \|z^n - z^\infty\|_{E_T} \\ &\leq \|z_0^n - z_0^\infty\|_1 + \frac{1}{2} \|z^n - z^\infty\|_{E_T}. \end{aligned}$$

Note that the right-hand side of the above inequality is independent of t and δ . By taking the supremum over $0 < \delta < 1, 0 < t < \frac{T(1-\delta)^\sigma}{2^\sigma - 1}$, we obtain

$$\|z^n - z^\infty\|_{E_T} \leq \|z_0^n - z_0^\infty\|_1 + \frac{1}{2} \|z^n - z^\infty\|_{E_T},$$

or

$$\|z^n - z^\infty\|_{E_T} \leq 2\|z_0^n - z_0^\infty\|_1.$$

This inequality holds true for any $n \geq N$ and completes the proof of Theorem 4.3. \square

5 Global Gevrey regularity and analyticity

We firstly introduce a lemma which is crucial to deal with the convection term of the system (1.1). The idea comes from [56], but we release the restriction on r .

Lemma 5.1. *Let $\delta \geq 0$, $\sigma \geq 1$ and $r > 1 + \frac{d}{2}$. Let $u \in G_{\sigma, r + \frac{1}{2\sigma}}^\delta(\mathbb{R}^d)$ and $w \in G_{\sigma, r + \frac{1}{2\sigma}}^\delta(\mathbb{R}^d)$. Then one has the estimate*

$$\begin{aligned} |\langle A^r e^{\delta A^{\frac{1}{\sigma}}} (u \cdot \nabla w), A^r e^{\delta A^{\frac{1}{\sigma}}} w \rangle| &\leq C \|A^r u\| \|A^r w\|^2 + C \delta (\|A^r e^{\delta A^{\frac{1}{\sigma}}} u\| \|A^{r + \frac{1}{2\sigma}} e^{\delta A^{\frac{1}{\sigma}}} w\|^2 + \\ &\quad + \|A^{r + \frac{1}{2\sigma}} e^{\delta A^{\frac{1}{\sigma}}} u\| \|A^r e^{\delta A^{\frac{1}{\sigma}}} w\| \|A^{r + \frac{1}{2\sigma}} e^{\delta A^{\frac{1}{\sigma}}} w\|), \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product of $L^2(\mathbb{R}^d)$, and $\|\cdot\| := \|\cdot\|_{L^2(\mathbb{R}^d)}$.

It is helpful to introduce two lemmas.

Lemma 5.2. *Let $r \geq 1$, $\sigma \geq 1$ and $\delta \geq 0$. Then for any real ξ, η , there holds*

$$\begin{aligned} &|(1 + \xi^2)^{\frac{r}{2}} e^{\delta(1 + \xi^2)^{\frac{1}{2\sigma}}} - (1 + \eta^2)^{\frac{r}{2}} e^{\delta(1 + \eta^2)^{\frac{1}{2\sigma}}}| \\ &\leq C_r |\xi - \eta| \left\{ (1 + |\xi - \eta|^2)^{\frac{r-1}{2}} + (1 + |\eta|^2)^{\frac{r-1}{2}} \right. \\ &\quad \left. + \delta \left[(1 + |\xi - \eta|^2)^{\frac{r-1}{2} + \frac{1}{2\sigma}} + (1 + |\eta|^2)^{\frac{r-1}{2} + \frac{1}{2\sigma}} \right] e^{\delta(1 + |\xi - \eta|^2)^{\frac{1}{2\sigma}}} e^{\delta(1 + |\eta|^2)^{\frac{1}{2\sigma}}} \right\}. \end{aligned} \quad (5.1)$$

Proof. Without loss of generality, take $\xi > \eta \geq 0$. Set $f(\theta) := (1 + \theta^2)^{\frac{r}{2}} e^{\delta(1 + \theta^2)^{\frac{1}{2\sigma}}}$. By the mean value theorem, we see

$$(1 + \xi^2)^{\frac{r}{2}} e^{\delta(1 + \xi^2)^{\frac{1}{2\sigma}}} - (1 + \eta^2)^{\frac{r}{2}} e^{\delta(1 + \eta^2)^{\frac{1}{2\sigma}}} \leq (\xi - \eta) \sup_{\theta \in [\eta, \xi]} |f'(\theta)|.$$

Computing f' and using the fact that $e^y \leq 1 + ye^y$ for $y \geq 0$ yield

$$\begin{aligned} f'(\theta) &= r\theta(1 + \theta^2)^{\frac{r}{2}-1} e^{\delta(1 + \theta^2)^{\frac{1}{2\sigma}}} + \frac{\delta}{\sigma} \theta(1 + \theta^2)^{\frac{r}{2} + \frac{1}{2\sigma} - 1} e^{\delta(1 + \theta^2)^{\frac{1}{2\sigma}}} \\ &\leq r(1 + \theta^2)^{\frac{r-1}{2}} \left[1 + \delta(1 + \theta^2)^{\frac{1}{2\sigma}} e^{\delta(1 + \theta^2)^{\frac{1}{2\sigma}}} \right] + \frac{\delta}{\sigma} (1 + \theta^2)^{\frac{r-1}{2} + \frac{1}{2\sigma}} e^{\delta(1 + \theta^2)^{\frac{1}{2\sigma}}} \\ &= r(1 + \theta^2)^{\frac{r-1}{2}} + \delta \left(r + \frac{1}{\sigma} \right) (1 + \theta^2)^{\frac{r-1}{2} + \frac{1}{2\sigma}} e^{\delta(1 + \theta^2)^{\frac{1}{2\sigma}}}. \end{aligned} \quad (5.2)$$

It is easy to verify that f' is a monotonically increasing function for $r \geq 1$, and hence the supremum in (5.2) is attained when $\theta = \xi$. For arbitrary non-negative ξ and η , we have

$$\begin{aligned} &|(1 + \xi^2)^{\frac{r}{2}} e^{\delta(1 + \xi^2)^{\frac{1}{2\sigma}}} - (1 + \eta^2)^{\frac{r}{2}} e^{\delta(1 + \eta^2)^{\frac{1}{2\sigma}}}| \\ &\leq |\xi - \eta| \left(r \left[(1 + \xi^2)^{\frac{r-1}{2}} + (1 + \eta^2)^{\frac{r-1}{2}} \right] \right. \\ &\quad \left. + \delta \left(r + \frac{1}{\sigma} \right) \left[(1 + \xi^2)^{\frac{r-1}{2} + \frac{1}{2\sigma}} e^{\delta(1 + \xi^2)^{\frac{1}{2\sigma}}} + (1 + \eta^2)^{\frac{r-1}{2} + \frac{1}{2\sigma}} e^{\delta(1 + \eta^2)^{\frac{1}{2\sigma}}} \right] \right). \end{aligned} \quad (5.3)$$

Note that for any $\xi, \eta, a, b \in \mathbb{R}$,

$$|\xi|^\rho \leq \begin{cases} |\xi - \eta|^\rho + |\eta|^\rho, & \text{when } \rho \in (0, 1], \\ 2^{\rho-1} (|\xi - \eta|^\rho + |\eta|^\rho), & \text{when } \rho > 1, \end{cases} \quad (5.4)$$

and

$$(1 + (a + b)^2)^{\frac{1}{2}} \leq (1 + a^2)^{\frac{1}{2}} + (1 + b^2)^{\frac{1}{2}}. \quad (5.5)$$

Since $\sigma \geq 1$, it follows that

$$(1 + \xi^2)^{\frac{1}{2\sigma}} \leq ((1 + |\xi - \eta|^2)^{\frac{1}{2}} + (1 + |\eta|^2)^{\frac{1}{2}})^{\frac{1}{\sigma}} \leq (1 + |\xi - \eta|^2)^{\frac{1}{2\sigma}} + (1 + |\eta|^2)^{\frac{1}{2\sigma}},$$

which leads to

$$e^{\delta(1+\xi^2)^{\frac{1}{2\sigma}}} \leq e^{\delta(1+|\xi-\eta|^2)^{\frac{1}{2\sigma}}} e^{\delta(1+\eta^2)^{\frac{1}{2\sigma}}} \quad (5.6)$$

Combining the estimates (5.3), (5.4) and (5.6), we obtain (5.1) and complete the proof of Lemma 5.2. \square

Now we introduce a lemma to deal with the interpolation of the Sobolev spaces and the Gervey spaces. The proof of this lemma is similar to that of Lemma 8 in [63].

Lemma 5.3. *For any $\delta \geq 0$, $\sigma \geq 1$, $l > 0$ and $r \in \mathbb{R}$, the following estimate holds true:*

$$\|u\|_{G_{\sigma,r}^\delta} \leq \sqrt{e} \|u\|_{H^r} + (2\delta)^{\frac{l}{2}} \|u\|_{G_{\sigma,r+\frac{l}{2\sigma}}^\delta}. \quad (5.7)$$

Proof. Noting that $e^y \leq e + y^l e^y$ for any $l > 0, y \geq 0$, we have

$$\begin{aligned} \|u\|_{G_{\sigma,r}^\delta}^2 &= \int (1 + |\xi|^2)^r e^{2\delta(1+|\xi|^2)^{\frac{1}{2\sigma}}} |\hat{u}(\xi)|^2 d\xi \\ &\leq \int (1 + |\xi|^2)^r (e + (2\delta)^l (1 + |\xi|^2)^{\frac{l}{2\sigma}} e^{2\delta(1+|\xi|^2)^{\frac{1}{2\sigma}}}) |\hat{u}(\xi)|^2 d\xi \\ &= e \int (1 + |\xi|^2)^r |u(\xi)|^2 d\xi + (2\delta)^l \int (1 + |\xi|^2)^{r+\frac{l}{2\sigma}} e^{2\delta(1+|\xi|^2)^{\frac{1}{2\sigma}}} |\hat{u}(\xi)|^2 d\xi \\ &= e \|u\|_{H^r}^2 + (2\delta)^l \|u\|_{G_{\sigma,r+\frac{l}{2\sigma}}^\delta}^2, \end{aligned}$$

which leads to (5.7). \square

Proof of Lemma 5.1. The idea comes from [56]. For simplicity, we only consider the case $d = 1$. Also, we omit the subscript \mathbb{R} and $d\xi d\eta$ of the integrands if there is no ambiguity. Write

$$\begin{aligned} \langle A^r e^{\delta A^{\frac{1}{\sigma}}} (u \partial_x w), A^r e^{\delta A^{\frac{1}{\sigma}}} w \rangle &= \langle A^r e^{\delta A^{\frac{1}{\sigma}}} (u \partial_x w), A^r e^{\delta A^{\frac{1}{\sigma}}} w \rangle - \langle u \partial_x A^r e^{\delta A^{\frac{1}{\sigma}}} w, A^r e^{\delta A^{\frac{1}{\sigma}}} w \rangle \\ &\quad + \langle u \partial_x A^r e^{\delta A^{\frac{1}{\sigma}}} w, A^r e^{\delta A^{\frac{1}{\sigma}}} w \rangle. \end{aligned}$$

Note that

$$\begin{aligned} |\langle u \partial_x A^r e^{\delta A^{\frac{1}{\sigma}}} w, A^r e^{\delta A^{\frac{1}{\sigma}}} w \rangle| &= \left| \int u (\partial_x A^r e^{\delta A^{\frac{1}{\sigma}}} w) A^r e^{\delta A^{\frac{1}{\sigma}}} w \right| \\ &= \left| -\frac{1}{2} \int u_x (A^r e^{\delta A^{\frac{1}{\sigma}}} w)^2 \right| \\ &\leq \frac{1}{2} \|u_x\|_{L^\infty} \|A^r e^{\delta A^{\frac{1}{\sigma}}} w\|^2 \\ &\leq C_r \|u\|_{H^r} (\|w\|_{H^r}^2 + \delta \|w\|_{G_{\sigma,r+\frac{l}{2\sigma}}^\delta}^2). \end{aligned} \quad (5.8)$$

The inequality on the last line is due to Lemma 5.3 with $l = 1$ and the embedding $H^{r-1} \hookrightarrow L^\infty$. Denote $\phi = A^r e^{\delta A^{\frac{1}{\sigma}}} w$. Next, we need to find the bound of

$$\begin{aligned} I &:= \langle A^r e^{\delta A^{\frac{1}{\sigma}}} (u \partial_x w), A^r e^{\delta A^{\frac{1}{\sigma}}} w \rangle - \langle u \partial_x A^r e^{\delta A^{\frac{1}{\sigma}}} w, A^r e^{\delta A^{\frac{1}{\sigma}}} w \rangle \\ &= \langle u \partial_x w, A^r e^{\delta A^{\frac{1}{\sigma}}} \phi \rangle - \langle u \partial_x A^r e^{\delta A^{\frac{1}{\sigma}}} w, \phi \rangle. \end{aligned}$$

By Planchel's identity, we have

$$\begin{aligned} \langle u \partial_x w, A^r e^{\delta A^{\frac{1}{\sigma}}} \phi \rangle &= \int_{\mathbb{R}} (u \partial_x w) (A^r e^{\delta A^{\frac{1}{\sigma}}} \phi) dx \\ &= i \int_{\mathbb{R}} \bar{\hat{\phi}}(\xi) (1 + |\xi|^2)^{\frac{r}{2}} e^{\delta(1+|\xi|^2)^{\frac{1}{2\sigma}}} \int_{\mathbb{R}} \hat{u}(\xi - \eta) \cdot \eta \hat{w}(\eta) d\eta d\xi \\ &= i \iint \bar{\hat{\phi}}(\xi) \hat{u}(\xi - \eta) \hat{w}(\eta) \eta (1 + |\xi|^2)^{\frac{r}{2}} e^{\delta(1+|\xi|^2)^{\frac{1}{2\sigma}}}, \\ \langle u \partial_x A^r e^{\delta A^{\frac{1}{\sigma}}} w, \phi \rangle &= i \iint \bar{\hat{\phi}}(\xi) \hat{u}(\xi - \eta) \hat{w}(\eta) \eta (1 + |\eta|^2)^{\frac{r}{2}} e^{\delta(1+|\eta|^2)^{\frac{1}{2\sigma}}}, \end{aligned}$$

where $\bar{\hat{\phi}}$ denotes the complex conjugate of the Fourier transformation of ϕ . Applying Lemma 5.2 to yield

$$\begin{aligned} |I| &\leq \iint |\bar{\hat{\phi}}(\xi)| |\hat{u}(\xi - \eta)| |\hat{w}(\eta)| |\eta| \left| (1 + |\xi|^2)^{\frac{r}{2}} e^{\delta(1+|\xi|^2)^{\frac{1}{2\sigma}}} - (1 + |\eta|^2)^{\frac{r}{2}} e^{\delta(1+|\eta|^2)^{\frac{1}{2\sigma}}} \right| \\ &\leq C_r \iint |\bar{\hat{\phi}}(\xi)| |\hat{u}(\xi - \eta)| |\hat{w}(\eta)| |\eta| |\xi - \eta| \left\{ (1 + |\xi - \eta|^2)^{\frac{r-1}{2}} + (1 + |\eta|^2)^{\frac{r-1}{2}} \right. \\ &\quad \left. + \delta \left[(1 + |\xi - \eta|^2)^{\frac{r-1}{2} + \frac{1}{2\sigma}} + (1 + |\eta|^2)^{\frac{r-1}{2} + \frac{1}{2\sigma}} \right] e^{\delta(1+|\xi-\eta|^2)^{\frac{1}{2\sigma}}} e^{\delta(1+|\eta|^2)^{\frac{1}{2\sigma}}} \right\}. \end{aligned} \quad (5.9)$$

By the definition of ϕ , and the fact that $e^y \leq 1 + ye^y$ for $y \geq 0$, we obtain

$$\begin{aligned} |\bar{\hat{\phi}}(\xi)| &= (1 + |\xi|^2)^{\frac{r}{2}} e^{\delta(1+|\xi|^2)^{\frac{1}{2\sigma}}} |\bar{\hat{w}}(\xi)| \\ &\leq (1 + |\xi|^2)^{\frac{r}{2}} (1 + \delta(1 + |\xi|^2)^{\frac{1}{\sigma}} e^{\delta(1+|\xi|^2)^{\frac{1}{2\sigma}}}) |\bar{\hat{w}}(\xi)| \\ &= (1 + |\xi|^2)^{\frac{r}{2}} |\bar{\hat{w}}(\xi)| + \delta(1 + |\xi|^2)^{\frac{1}{\sigma}} |\bar{\hat{\phi}}(\xi)|. \end{aligned} \quad (5.10)$$

Combining the estimate (5.9) and (5.10), we can divide (5.9) into four parts

$$\begin{aligned} |I| &\leq C_r \iint (1 + |\xi|^2)^{\frac{r}{2}} |\bar{\hat{w}}(\xi)| |\hat{u}(\xi - \eta)| |\hat{w}(\eta)| |\eta| |\xi - \eta| \left\{ (1 + |\xi - \eta|^2)^{\frac{r-1}{2}} + (1 + |\eta|^2)^{\frac{r-1}{2}} \right\} d\eta d\xi \\ &\quad + C_r \delta \iint |\bar{\hat{\phi}}(\xi)| |\hat{u}(\xi - \eta)| |\hat{w}(\eta)| |\eta| |\xi - \eta| \\ &\quad \times \left\{ (1 + |\xi - \eta|^2)^{\frac{r-1}{2} + \frac{1}{2\sigma}} + (1 + |\eta|^2)^{\frac{r-1}{2} + \frac{1}{2\sigma}} \right\} e^{\delta(1+|\xi-\eta|^2)^{\frac{1}{2\sigma}}} e^{\delta(1+|\eta|^2)^{\frac{1}{2\sigma}}} d\eta d\xi \\ &:= C_r (I_1 + I_2) + C_r \delta \cdot (I_3 + I_4). \end{aligned} \quad (5.11)$$

Firstly we consider the term

$$\begin{aligned} I_1 &= \iint (1 + |\xi|^2)^{\frac{r}{2}} |\bar{\hat{w}}(\xi)| |\hat{u}(\xi - \eta)| |\hat{w}(\eta)| |\eta| |\xi - \eta| (1 + |\xi - \eta|^2)^{\frac{r-1}{2}} d\eta d\xi \\ &\leq \int |\hat{w}(\eta)| (1 + |\eta|^2)^{\frac{1}{2}} \left(\int (1 + |\xi|^2)^{\frac{r}{2}} |\bar{\hat{w}}(\xi)| (1 + |\xi - \eta|^2)^{\frac{r}{2}} |\hat{u}(\xi - \eta)| d\xi \right) d\eta \end{aligned}$$

$$\begin{aligned}
 &\leq \int |\hat{w}(\eta)| (1 + |\eta|^2)^{\frac{r}{2}} (1 + |\eta|^2)^{\frac{1}{2} - \frac{r}{2}} d\eta \|A^r u\| \|A^r w\| \\
 &\leq C_r \|A^r u\| \|A^r w\|^2.
 \end{aligned} \tag{5.12}$$

Similarly, after the transformation $\xi' = \xi, \eta' = \xi - \eta$, we have

$$I_2 = \iint (1 + |\xi|^2)^{\frac{r}{2}} |\hat{w}(\xi)| |\hat{u}(\xi - \eta)| |\hat{w}(\eta)| |\eta| |\xi - \eta| (1 + |\eta|^2)^{\frac{r-1}{2}} \leq C_r \|A^r u\| \|A^r w\|^2. \tag{5.13}$$

Next, we consider the term I_3 in (5.11). Taking advantage of (5.4) and (5.5), we obtain

$$(1 + |\xi - \eta|^2)^{\frac{1}{4\sigma}} \leq ((1 + |\xi|^2)^{\frac{1}{2}} + (1 + |\eta|^2)^{\frac{1}{2}})^{\frac{1}{2\sigma}} \leq (1 + |\xi|^2)^{\frac{1}{4\sigma}} + (1 + |\eta|^2)^{\frac{1}{4\sigma}},$$

and hence

$$\begin{aligned}
 I_3 &= \iint |\hat{\phi}(\xi)| |\hat{u}(\xi - \eta)| |\hat{w}(\eta)| |\eta| |\xi - \eta| (1 + |\xi - \eta|^2)^{\frac{r-1}{2} + \frac{1}{2\sigma}} e^{\delta(1+|\xi-\eta|^2)\frac{1}{2\sigma}} e^{\delta(1+|\eta|^2)\frac{1}{2\sigma}} \\
 &\leq \iint |\hat{\phi}(\xi)| |\hat{u}(\xi - \eta)| |\hat{w}(\eta)| (1 + |\eta|^2)^{\frac{1}{2}} (1 + |\xi - \eta|^2)^{\frac{r}{2} + \frac{1}{4\sigma}} \\
 &\quad \times \left[(1 + |\xi|^2)^{\frac{1}{4\sigma}} + (1 + |\eta|^2)^{\frac{1}{4\sigma}} \right] e^{\delta(1+|\xi-\eta|^2)\frac{1}{2\sigma}} e^{\delta(1+|\eta|^2)\frac{1}{2\sigma}} \\
 &:= I_{31} + I_{32}.
 \end{aligned} \tag{5.14}$$

It then follows that

$$\begin{aligned}
 I_{31} &= \iint |\hat{\phi}(\xi)| (1 + |\xi|^2)^{\frac{1}{4\sigma}} |\hat{u}(\xi - \eta)| (1 + |\xi - \eta|^2)^{\frac{r}{2} + \frac{1}{4\sigma}} e^{\delta(1+|\xi-\eta|^2)\frac{1}{2\sigma}} |\hat{w}(\eta)| (1 + |\eta|^2)^{\frac{1}{2}} e^{\delta(1+|\eta|^2)\frac{1}{2\sigma}} d\xi d\eta \\
 &\leq \int |\hat{w}(\eta)| (1 + |\eta|^2)^{\frac{r}{2}} e^{\delta(1+|\eta|^2)\frac{1}{2\sigma}} (1 + |\eta|^2)^{\frac{1}{2} - \frac{r}{2}} d\eta \|A^{r+\frac{1}{2\sigma}} e^{\delta A^{\frac{1}{\sigma}}} u\| \|A^{\frac{1}{2\sigma}} \phi\| \\
 &\leq \left(\int (1 + |\eta|^2)^{1-r} d\eta \right)^{\frac{1}{2}} \|A^r e^{\delta A^{\frac{1}{\sigma}}} w\| \|A^{r+\frac{1}{2\sigma}} e^{\delta A^{\frac{1}{\sigma}}} u\| \|A^{r+\frac{1}{2\sigma}} e^{\delta A^{\frac{1}{\sigma}}} w\| \\
 &= C_r \|A^r e^{\delta A^{\frac{1}{\sigma}}} w\| \|A^{r+\frac{1}{2\sigma}} e^{\delta A^{\frac{1}{\sigma}}} u\| \|A^{r+\frac{1}{2\sigma}} e^{\delta A^{\frac{1}{\sigma}}} w\|,
 \end{aligned} \tag{5.15}$$

and

$$\begin{aligned}
 I_{32} &= \iint |\hat{\phi}(\xi)| |\hat{u}(\xi - \eta)| (1 + |\xi - \eta|^2)^{\frac{r}{2} + \frac{1}{4\sigma}} e^{\delta(1+|\xi-\eta|^2)\frac{1}{2\sigma}} |\hat{w}(\eta)| (1 + |\eta|^2)^{\frac{1}{2} + \frac{1}{4\sigma}} e^{\delta(1+|\eta|^2)\frac{1}{2\sigma}} d\xi d\eta \\
 &\leq \int |\hat{w}(\eta)| (1 + |\eta|^2)^{\frac{r}{2} + \frac{1}{4\sigma}} e^{\delta(1+|\eta|^2)\frac{1}{2\sigma}} (1 + |\eta|^2)^{\frac{1}{2} - \frac{r}{2}} d\eta \|A^{r+\frac{1}{2\sigma}} e^{\delta A^{\frac{1}{\sigma}}} u\| \|\phi\| \\
 &\leq \left(\int (1 + |\eta|^2)^{1-r} d\eta \right)^{\frac{1}{2}} \|A^{r+\frac{1}{2\sigma}} e^{\delta A^{\frac{1}{\sigma}}} w\| \|A^{r+\frac{1}{2\sigma}} e^{\delta A^{\frac{1}{\sigma}}} u\| \|A^r e^{\delta A^{\frac{1}{\sigma}}} w\| \\
 &= C_r \|A^r e^{\delta A^{\frac{1}{\sigma}}} w\| \|A^{r+\frac{1}{2\sigma}} e^{\delta A^{\frac{1}{\sigma}}} u\| \|A^{r+\frac{1}{2\sigma}} e^{\delta A^{\frac{1}{\sigma}}} w\|.
 \end{aligned} \tag{5.16}$$

Plugging (5.15) and (5.16) into (5.14) yields

$$I_3 \leq C_r \|A^r e^{\delta A^{\frac{1}{\sigma}}} w\| \|A^{r+\frac{1}{2\sigma}} e^{\delta A^{\frac{1}{\sigma}}} u\| \|A^{r+\frac{1}{2\sigma}} e^{\delta A^{\frac{1}{\sigma}}} w\|. \tag{5.17}$$

Similarly, by the transformation $\xi' = \xi, \eta' = \xi - \eta$, we obtain

$$\begin{aligned}
 I_4 &= \iint |\hat{\phi}(\xi)| |\hat{u}(\xi - \eta)| |\hat{w}(\eta)| |\eta| |\xi - \eta| (1 + |\eta|^2)^{\frac{r-1}{2} + \frac{1}{2\sigma}} e^{\delta(1+|\xi-\eta|^2)\frac{1}{2\sigma}} e^{\delta(1+|\eta|^2)\frac{1}{2\sigma}} \\
 &\leq C_r \|A^r e^{\delta A^{\frac{1}{\sigma}}} u\| \|A^{r+\frac{1}{2\sigma}} e^{\delta A^{\frac{1}{\sigma}}} w\|^2.
 \end{aligned} \tag{5.18}$$

According to (5.8), (5.12), (5.13), (5.17) and (5.18), we complete the proof of Lemma 5.1. \square

Now we can state a main theorem of this section, which implies is the global Gevrey regularity and analyticity result of the equation (1.2).

Theorem 5.4. *Assuem $s = [s] \geq 2, a = 2$ and $\sigma \geq 1$. Let u_0 be in $G_\sigma(\mathbb{R})$. Then there exists a unique global solution u of (1.2) in Gevrey class σ , namely, for any $t \geq 0$, $u(t, \cdot)$ is of Gevrey class σ .*

Proof. Here we only derive the global a priori bounds on u in the time-dependent space $G_{\sigma,q}^{\delta(t)}$. One can use Fourier-Galerkin approximating method to construct local solutions in $G_{\sigma,q}^{\delta(t)}$, and globalize the result by the later estimate (5.25). In the following, we will find a $\delta(t)$ to keep the solution of Gervey class σ . Note that if $u_0 \in G_{\sigma,q}^{\delta_0}$ then $u_0 \in G_{\sigma,\infty}^{\delta_0-\varepsilon}$ for any $\varepsilon > 0$. Without loss of generality, we may assume that $q > s + \frac{1}{2}$ and $u_0 \in G_{\sigma,q}^1(\mathbb{R})$. Since for every $t \in [0, T)$

$$u_t = -uu_x + K(u, u),$$

we then have

$$\begin{aligned} \frac{d}{dt} \|u\|_{G_{\sigma,q}^{\delta(t)}}^2 &= \frac{d}{dt} \int (1 + |\xi|^2)^q e^{2\delta(t)(1+|\xi|^2)^{\frac{1}{2\sigma}}} \hat{u}(\xi) \bar{\hat{u}}(\xi) d\xi \\ &= 2\dot{\delta}(t) \int (1 + |\xi|^2)^{q+\frac{1}{2\sigma}} e^{2\delta(t)(1+|\xi|^2)^{\frac{1}{2\sigma}}} \hat{u}(\xi) \bar{\hat{u}}(\xi) d\xi \\ &\quad + 2\Re \int (1 + |\xi|^2)^q e^{2\delta(t)(1+|\xi|^2)^{\frac{1}{2\sigma}}} \left(-\widehat{uu_x}(\xi) + \widehat{K(u, u)}(\xi) \right) \bar{\hat{u}}(\xi) d\xi, \end{aligned} \quad (5.19)$$

where \Re denotes the real part of a complex number. Applying Lemma 5.1 to obtain

$$\begin{aligned} \left| \int (1 + |\xi|^2)^q e^{2\delta(t)(1+|\xi|^2)^{\frac{1}{2\sigma}}} \widehat{uu_x}(\xi) \bar{\hat{u}}(\xi) d\xi \right| &= |\langle A^q e^{\delta A^{\frac{1}{\sigma}}}(u \partial_x u), A^q e^{\delta A^{\frac{1}{\sigma}}} u \rangle| \\ &\leq C_q \left(\|u\|_{H^q}^3 + \delta \|u\|_{G_{\sigma,q}^{\delta}} \|u\|_{G_{\sigma,q+\frac{1}{2\sigma}}^{\delta}}^2 \right). \end{aligned} \quad (5.20)$$

To control the term involved $K(u, u)$, for simplicity, we only control the first and the last term of the highest order in (3.1), namely,

$$\begin{aligned} &\left| \int (1 + |\xi|^2)^q e^{2\delta(t)(1+|\xi|^2)^{\frac{1}{2\sigma}}} \mathcal{F}((1 - \partial_x^2)^{-s} \partial_x^{2s-1}(u_x^2))(\xi) \bar{\hat{u}}(\xi) d\xi \right| \\ &= |\langle A^q e^{\delta A^{\frac{1}{\sigma}}} ((1 - \partial_x^2)^{-s} \partial_x^{2s-1}(u_x^2)), A^q e^{\delta A^{\frac{1}{\sigma}}} u \rangle| \\ &\leq \|A^{q-2s} e^{\delta A^{\frac{1}{\sigma}}} \partial_x^{2s-1}(u_x^2)\|_{L^2} \|A^q e^{\delta A^{\frac{1}{\sigma}}} u\|_{L^2} \\ &\leq \|u_x^2\|_{G_{\sigma,q-1}^{\delta}} \|u\|_{G_{\sigma,q}^{\delta}} \\ &\leq C_q \|u\|_{G_{\sigma,q}^{\delta}}^3, \end{aligned}$$

and

$$\begin{aligned} &\left| \int (1 + |\xi|^2)^q e^{2\delta(t)(1+|\xi|^2)^{\frac{1}{2\sigma}}} \mathcal{F}((1 - \partial_x^2)^{-s} \partial_x [(\partial_x^s u)^2])(\xi) \bar{\hat{u}}(\xi) d\xi \right| \\ &= |\langle A^q e^{\delta A^{\frac{1}{\sigma}}} ((1 - \partial_x^2)^{-s} \partial_x [(\partial_x^s u)^2]), A^q e^{\delta A^{\frac{1}{\sigma}}} u \rangle| \\ &\leq \|A^{q-2s} e^{\delta A^{\frac{1}{\sigma}}} \partial_x [(\partial_x^s u)^2]\|_{L^2} \|A^q e^{\delta A^{\frac{1}{\sigma}}} u\|_{L^2} \\ &\leq \|(\partial_x^s u)^2\|_{G_{\sigma,q-2s+1}^{\delta}} \|u\|_{G_{\sigma,q}^{\delta}} \\ &\leq \|(\partial_x^s u)^2\|_{G_{\sigma,q-s}^{\delta}} \|u\|_{G_{\sigma,q}^{\delta}} \\ &\leq C_{q,s} \|u\|_{G_{\sigma,q}^{\delta}}^3. \end{aligned}$$

Therefore, using Lemma 5.3 with $l = \frac{2}{3}$, we have

$$\begin{aligned} \left| \int (1 + |\xi|^2)^q e^{2\delta(t)(1+|\xi|^2)^{\frac{1}{2\sigma}}} \widehat{K(u, u)}(\xi) \bar{u}(\xi) d\xi \right| &\leq C_{q,s} \|u\|_{G_{\sigma,q}^\delta}^3 \\ &\leq C_{q,s} \left(\|u\|_{H^q}^3 + \delta \|e^{\delta A^{\frac{1}{\sigma}}} u\|_{H^{q+\frac{1}{3\sigma}}}^3 \right) \\ &\leq C_{q,s} \left(\|u\|_{H^q}^3 + \delta \|u\|_{G_{\sigma,q}^\delta} \|u\|_{G_{\sigma,q+\frac{1}{2\sigma}}^\delta}^2 \right). \end{aligned} \quad (5.21)$$

The inequality on the last line is due to the Sobolev interpolation inequality

$$\|f\|_{H^{q+\frac{1}{3\sigma}}} \leq \|f\|_{H^q}^{\frac{1}{3}} \|f\|_{H^{q+\frac{1}{2\sigma}}}^{\frac{2}{3}}.$$

Plugging estimates (5.20) and (5.21) into (5.19) yields

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_{G_{\sigma,q}^{\delta(t)}}^2 \leq (\dot{\delta}(t) + C\delta(t) \|u(t)\|_{G_{\sigma,q}^{\delta(t)}}) \|u(t)\|_{G_{\sigma,q+\frac{1}{2\sigma}}^{\delta(t)}}^2 + C \|u(t)\|_{H^q}^3. \quad (5.22)$$

Theorem 2.12 guarantees the existence of global classical solution $u \in \mathcal{C}([0, \infty); H^q)$, whence $\theta(t) := \|u(t)\|_{H^q} \in \mathcal{C}(\mathbb{R}^+)$. To ensure that the first term on the right-hand side of (5.22) is negligible, we can set

$$\dot{\delta}(t) = -Ch(t)\delta(t), \quad (5.23)$$

or

$$\delta(t) = \delta_0 \exp \left(-C \int_0^t h(t') dt' \right), \quad (5.24)$$

where $0 < \delta_0 < 1$, and

$$h^2(t) := 2\|u_0\|_{G_{\sigma,q}^{\delta_0}}^2 + 2C \int_0^t \theta^3(t') dt'.$$

Thus, we can use the bootstrap argument to ensure that, for any $t \in [0, \infty)$,

$$\|u(t)\|_{G_{\sigma,q}^{\delta(t)}}^2 \leq h^2(t) = 2\|u_0\|_{G_{\sigma,q}^{\delta_0}}^2 + 2C \int_0^t \theta^3(t') dt'. \quad (5.25)$$

This completes the proof of Theorem 5.4.

Remark 5.5. Let $\sigma = 1$, $a = 2$. According to the Theorem 3.1, we can obtain the global analyticity in time as well as space. Namely, for real analytic $u_0 \in G_1$, there exists a unique analytic solution to (1.2), i.e.,

$$u \in C^\omega([0, \infty) \times \mathbb{R}).$$

Moreover, for every $t \geq 0$, the solution $w(t)$ lies in Gevrey class G_1 .

Remark 5.6. When $s > \frac{3}{2}$, $a = 2$, according to Lemma 2.13, we can also have the global analyticity (or Gevrey regularity) solution if the initial data u_0 is in G_1 (or G_σ with $\sigma > 1$). However, we will not prospect the global analyticity or Gevrey regularity when $s = 1$, $a = 2$ unless we restrict the sign condition on u_0 such that $m_0 := u_0 - u_{0,xx}$ does not change sign. □

Similarly, we can obtain the following global Gevrey regularity and analyticity result of the two-component system (1.1), which relies on the global strong solution results—Lemma 2.14–2.15.

Theorem 5.7. *Let $a = 2$, $\kappa \geq 0$, $s > \frac{3}{2}$, $\sigma \geq 1$. Assume that $(u_0, \rho_0) \in G_\sigma(\mathbb{R}) \times G_\sigma(\mathbb{R})$. There exists a unique global solution (u, ρ) of (1.1) in Gevrey class $G_\sigma \times G_\sigma$, namely, for any $t \geq 0$, $(u(t, \cdot))$ and $\rho(t, \cdot)$ are both in Gevrey class G_σ .*

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References

- [1] A. Bressan and A. Constantin, *Global conservative solutions of the Camassa-Holm equation*, Arch. Rat. Mech. Anal., **183** (2007), 215–239.
- [2] A. Bressan and A. Constantin, *Global dissipative solutions of the Camassa-Holm equation*, Anal. Appl., **5** (2007), 1–27.
- [3] M. S. Baouendi and C. Goulaouic, *Remarks on the abstract form of nonlinear Cauchy-Kovalevsky theorems*, Communications in Partial Differential Equations, **2** (1977), 1151–1162.
- [4] R. F. Barostichi, A. Himonas and G. Petronilho, *Autonomous Ovsyannikov theorem and applications to nonlocal evolution equations and systems*, J. Funct. Anal. **270** (2016), 330–358.
- [5] R. Camassa and D. D. Holm, *An integrable shallow water equation with peaked solitons*, Phys. Rev. Lett., **71** (1993), 1661–1664.
- [6] R. Camassa and D. D. Holm, *A new integrable shallow water equation*, Adv. Appl. Mech., **31** (1994), 1–33.
- [7] M. Chen, S-Q. Liu and Y. Zhang, *A two-component generalization of the Camassa-Holm equation and its solutions*, Lett. Math. Phys., **75** (2006), 1–15.
- [8] G. M. Coclite and K. H. Karlsen, *On the well-posedness of the Degasperis-Procesi equation*, J. Funct. Anal., **233** (2006), 60–91.
- [9] G. M. Coclite, H. Holden and K. H. Karlsen, *Well-posedness of higher-order Camassa-Holm equations*, J. Differential Equations, **246** (2009), 929–963.
- [10] A. Constantin, *On the inverse spectral problem for the Camassa-Holm equation*, J. Funct. Anal., **155** (1998), 352–363.
- [11] A. Constantin, *Existence of permanent and breaking waves for a shallow water equation: a geometric approach*, Ann. Inst. Fourier (Grenoble), **50** (2000), 321–362.
- [12] A. Constantin, *On the scattering problem for the Camassa-Holm equation*, Proc. Roy. Soc. London A, **457** (2001), 953–970.
- [13] A. Constantin, *Finite propagation speed for the Camassa-Holm equation*, J. Math. Phys., **46** (2005), 023506.
- [14] A. Constantin and J. Escher, *Wave breaking for nonlinear nonlocal shallow water equations*, Acta Math., **181** (1998), 229–243.
- [15] A. Constantin and J. Escher, *Global existence and blow-up for a shallow water equation*, Annali Sc. Norm. Sup. Pisa, **26** (1998), 303–328.

-
- [16] A. Constantin and J. Escher, *Well-posedness, global existence and blowup phenomena for a periodic quasi-linear hyperbolic equation*, Comm. Pure Appl. Math., **51** (1998), 475–504.
 - [17] A. Constantin and J. Escher, *Global weak solutions for a shallow water equation*, Indiana Uni. Math. J., **47** (1998), 1527–1545.
 - [18] A. Constantin and J. Escher, *On the blow-up rate and the blow-up set of breaking waves for a shallow water equation*, Math. Z., **233** (2000), 75–91.
 - [19] A. Constantin, V. S. Gerdjikov and R. Ivanov, *Inverse scattering transform for the Camassa-Holm equation*, Inverse Problems, **22** (2006), 2197–2207.
 - [20] A. Constantin and R. Ivanov, *On an integrable two-component Camassa-Holm shallow water system*, Phys. Lett. A, **372** (2008), 7129–7132.
 - [21] A. Constantin, R. Ivanov and J. Lenells, *Inverse scattering transform for the Degasperis-Procesi equation*, Nonlinearity, **23** (2010), 2559–2575.
 - [22] A. Constantin and B. Kolev, *Geodesic flow on the diffeomorphism group of the circle*, Comment. Math. Helvetici, **78** (2003), 787–804.
 - [23] A. Constantin, T. Kappeler, B. Kolev and B. Topalov, *On geodesic exponential maps of the Virasoro group*, Ann. Glob. Anal. Geom., **31** (2007), 155–180.
 - [24] A. Constantin and H. P. McKean, *A shallow water equation on the circle*, Comm. Pure Appl. Math., **52** (1999), 949–982.
 - [25] A. Constantin and L. Molinet, *Global weak solutions for a shallow water equation*, Comm. Math. Phys., **211** (2000), 45–61.
 - [26] R. Danchin, *A few remarks on the Camassa-Holm equation*, Differential and Integral Equations **14** (2001), 953–988.
 - [27] A. Degasperis and M. Procesi, *Asymptotic integrability*, Symmetry and perturbation theory (Rome 1998), pp. 23–37. World Sci. Publ., River Edge, NJ, 1999.
 - [28] J. Escher, *Non-metric two-component Euler equations on the circle*, Monatsh. Math., **167** (2012), 449–459.
 - [29] J. Escher, D. Henry, B. Kolev and T. Lyons, *Two-component equations modelling water waves with constant vorticity*, Ann. Mat. Pura Appl. (4), **195** (2016), 249–271.
 - [30] J. Escher and B. Kolev, *The Degasperis-Procesi equation as a non-metric Euler equation*, Math. Z., **269** (2011), 1137–1153.
 - [31] J. Escher, M. Kohlmann and J. Lenells, *The geometry of the two-component Camassa-Holm and Degasperis-Procesi equations*, J. Geom. Phys., **61** (2011), 436–452.
 - [32] J. Escher, O. Lechtenfeld and Z. Yin, *Well-posedness and blow-up phenomena for the 2-component Camassa-Holm equation*, Discrete Contin. Dyn. Syst., **19** (2007), 493–513.
 - [33] J. Escher, Y. Liu and Z. Yin, *Global weak solutions and blow-up structure for the Degasperis-Procesi equation*, J. Funct. Anal., **241** (2006), 457–485.
 - [34] J. Escher, Y. Liu and Z. Yin, *Shock waves and blow-up phenomena for the periodic Degasperis-Procesi equation*, Indiana Univ. Math. J., **56** (2007), 87–177.

-
- [35] J. Escher and T. Lyons, *Two-component higher order Camassa-Holm systems with fractional inertia operator: a geometric approach*, J. Geom. Mech. **7** (2015), 281C293.
 - [36] J. Escher and Z. Yin, *Well-posedness, blow-up phenomena, and global solutions for the b-equation*, J. Reine Angew. Math., **624** (2008), 51–80.
 - [37] J. Escher and Z. Yin, *Initial boundary value problems of the Camassa-Holm equation*, Commun. Partial Differential Equations, **33** (2008), 377–395.
 - [38] J. Escher and Z. Yin, *Initial boundary value problems for nonlinear dispersive wave equations*, J. Funct. Anal., **256** (2009), 479–508.
 - [39] C. Foias and R. Temam, *Gevrey class regularity for the solutions of the Navier-Stokes equations*, J. Funct. Anal., **87** (1989), 359–369.
 - [40] C. Guan ,H. He, Z. Yin, *Well-posedness, blow-up phenomena and persistence properties for a two-component water wave system*, Nonlinear Anal. Real World Appl. , **25** (2015), 219C237.
 - [41] C. Guan, H. Karlsen and Z. Yin, *Well-posedness and blow-up phenomena for a modified two-component Camassa-Holm equation*, Contemporary Mathematics, **526** (2010), 199–220.
 - [42] C. Guan and Z. Yin, *Global existence and blow-up phenomena for an integrable two-component Camassa-Holm shallow water system*, J. Differential Equations, **248** (2010), 2003–2014.
 - [43] C. Guan and Z. Yin, *Global weak solutions for a two-component Camassa-Holm shallow water system*, J. Funct. Anal., **260** (2011), 1132–1154.
 - [44] C. Guan and Z. Yin, *Global weak solutions for a modified two-component Camassa-Holm equation*, Ann. I. H. Poincaré-AN, **28** (2011), 623–641.
 - [45] C. Guan and Z. Yin, *On the existence of global weak solutions to an integrable two-component CamassaCHolm shallow-water system*, Proc. Edinb. Math. Sic., **56** (2013), 755–775.
 - [46] C. Guan and Z. Yin, *On the global weak solutions for a modified two-component Camassa-Holm equation*, Math. Nach., **286** (2013), 1287–1304.
 - [47] G. Gui, Y. Liu and L. Tian, *Global existence and blow-up phenomena for the peakon b-family of equations*, Indiana Univ. Math. J., **57** (2008), 1209–1234.
 - [48] D. Henry, *Compactly supported solutions of a family of nonlinear partial differential equations*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., **15** (2008), 145–150.
 - [49] D. Henry, *Persistence properties for a family of nonlinear partial differential equations*, Nonlinear Anal., **70** (2009), 1565–1573.
 - [50] A. Himonas and G. Misiolek, *Analyticity of the Cauchy problem for an integrable evolution equation*, Mathematische Annalen, **327** (2003) , 575–584.
 - [51] A. Himonas, G. Misiolek, G. Ponce and Y. Zhou, *Persistence properties and unique continuation of solutions of the Camassa-Holm equation*, Comm. Math. Phys., **27** (2007), 511–522.
 - [52] H. He and Z. Yin, *Well-posedness and analytic solutions of a two-component water wave equation*, J. Math. Anal. Appl. **434** (2016), 353C375.
 - [53] H. He and Z. Yin, *On the Cauchy problem for a generalized two-component shallow water wave system with fractional higher-order inertia operators*, Discrete Contin. Dyn. Syst., to appear.

-
- [54] R. Ivanov, *Water waves and integrability*, Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., **365** (2007), 2267–2280.
 - [55] J. Lenells, *Traveling wave solutions of the Degasperis-Procesi equation*, J. Math. Anal. Appl., **306** (2005), 72–82.
 - [56] C. D. Levermore and M. Oliver, *Analyticity of solutions for a generalized Euler equation*, J. Differential Equations, **133** (1997), 321–339.
 - [57] Y. Li and P. Olver, *Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation*, J. Differential Equations, **162** (2000), 27–63.
 - [58] Y. Liu and Z. Yin, *Global existence and blow-up phenomena for the Degasperis-Procesi equation*, Commun. Math. Phys., **267** (2006), 801–820.
 - [59] H. Lundmark and J. Szmigielski, *Multi-peakon solutions of the Degasperis-Procesi equation*, Inverse Problems, **19** (2003), 1241–1245.
 - [60] W. Luo and Z. Yin, *Gevrey regularity and analyticity for Camassa-Holm type systems*, Preprint arXiv: /1507.05250.pdf.
 - [61] Y. Matsuno, *Multisoliton solutions of the Degasperis-Procesi equation and their peakon limit*, Inverse Problems, **21** (2005), 1553–1570.
 - [62] C. Mu, S. Zhou and R. Zeng, *Well-posedness and blow-up phenomena for a higher order shallow water equation*, J. Differential Equations, **251** (2011), 3488–3499.
 - [63] M. Oliver and E. S. Titi, *Remark on the rate of decay of higher order derivatives for solutions to the Navier-Stokes equations in \mathbb{R}^n* , J. Funct. Anal., **172** (2000), 1–18.
 - [64] L.V. Ovsyannikov, *Singular operators in Banach spaces scales*, Doklady Akademii Nauk SSSR, **163** (1965) 819–822.
 - [65] L.V. Ovsyannikov, *Non-local Cauchy problems in fluid dynamics*, Actes du Congrès International des Mathématiciens **3** (1970), 137–142.
 - [66] L.V. Ovsyannikov, *A nonlinear Cauchy problem in a scale of Banach spaces*, Doklady Akademii Nauk SSSR **200**(1971), 789–792.
 - [67] G. Rodriguez-Blanco, *On the Cauchy problem for the Camassa-Holm equation*, Nonlinear Anal., **46** (2001), 309–327.
 - [68] W. Tan and Z. Yin, *Global conservative solutions of a modified two-component CamassaCHolm shallow water system*, J. Differential Equations, **251** (2011), 3558–3582.
 - [69] W. Tan and Z. Yin, *Global periodic conservative solutions of a periodic modified two-component CamassaCHolm equation*, J. Funct. Anal., **261** (2011), 1204–1226.
 - [70] Z. Xin and P. Zhang, *On the weak solutions to a shallow water equation*, Comm. Pure Appl. Math., **53** (2000), 1411–1433.
 - [71] K. Yan and Z. Yin, *Analytic solutions of the Cauchy problem for two-component shallow water systems*, Mathematische Zeitschrift, **269** (2011), 1113–1127.
 - [72] Z. Yin, *Well-posedness, blowup, and global existence for an integrable shallow water equation*, Discrete Continuous Dynam. Systems, **10** (2004), 393–411.

- [73] Z. Yin, *On the Cauchy problem for an integrable equation with peakon solutions*, III. J. Math., **47** (2003), 649–666.
- [74] Z. Yin, *Global existence for a new periodic integrable equation*, J. Math. Anal. Appl., **49** (2003), 129–139.
- [75] Z. Yin, *Global weak solutions for a new periodic integrable equation with peakon solutions*, J. Funct. Anal., **212** (2004), 182–194.
- [76] Z. Yin, *Global solutions to a new integrable equation with peakons*, Indiana Univ. Math. J., **53** (2004), 1189–1210.
- [77] S. Zhang and Z. Yin, *Global solutions and blow-up phenomena for the periodic b-equation*, J. Lond. Math. Soc., **82** (2010), 482–500.